On recursion operators for elliptic models

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Abstract. New quasilocal recursion and Hamiltonian operators for the Krichever-Novikov and the Landau-Lifshitz equations are found. It is shown that the associative algebra of quasilocal recursion operators for these models is generated by a couple of operators related by an elliptic curve equation. A theoretical explanation of this fact for the Landau-Lifshitz equation is given in terms of multiplicators of the corresponding Lax structure.

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Introduction

One of the main algebraic structures related to $1+1$-dimensional integrable PDE of the form

$$u_t = F(u, u_x, \dots, u_n)\,, \qquad n \geq 2, \qquad u_i = \partial_x^i u(x,t) \tag{0.1}$$

is an infinite hierarchy of commuting flows or, the same, (generalized) symmetries [1]

$$u_{t_i} = G_i(u, ..., u_{m_i})\,. \tag{0.2}$$

We identify the symmetry (0.2) with its right hand side G_i. The symmetry G satisfies the equation

$$D_t(G) - F_*(G) = 0, \tag{0.3}$$

where D_t stands for the derivation in virtue of (0.1) and F_* denotes the Fréchet derivative of F:

$$F_* = \sum_{i=0}^{n} \frac{\partial^i F}{\partial u_i} D_x^i.$$

The dual objects for symmetries are cosymmetries which satisfy the equation

$$D_t(g) + F_*^t(g) = 0,$$

where F_*^t is the differential operator adjoint to F_*. The product $g\,G$ of any cosymmetry g and symmetry G is a total x-derivative. It is well known [1] that for any conserved density ρ the variational derivative $\frac{\delta\rho}{\delta u}$ is a cosymmetry.

The simplest symmetry for any equation (0.1) is u_x. The usual way to get other symmetries is to act to u_x by a recursion operator $\mathcal{R}$. By definition, the recursion operator is a ratio of two differential operators that satisfies the identity

$$[D_t - F_*,\, \mathcal{R}] = \mathcal{R}_t - [F_*,\, \mathcal{R}] = 0. \tag{0.4}$$

It follows from (0.3) and (0.4) that for any symmetry G the expression $\mathcal{R}(G)$ is a symmetry as well.

Most of known recursion operators have the following special form

$$\mathcal{R} = R + \sum_{i=1}^{k} G_i D_x^{-1} g_i, \tag{0.5}$$

where R is a differential operator, G_i and g_i are some fixed symmetries and cosymmetries common for all members of the hierarchy. For all known examples the cosymmetries g_i are variational derivatives of conserved densities. Applying such operator to any symmetry we get a local expression, (i.e. a function of finite number of variables $u, u_x, \dots u_i, \dots$) since the product of any symmetry and cosymmetry belongs to $Im\, D_x$. Moreover, a different choice of integration

constants gives rise to an additional linear combination of the symmetries $G_1, \ldots, G_k$. Probably for the first time ansatz (0.5) was used in [2]. We call recursion operators (0.5) *quasilocal.*

Most of known integrable equations (0.1) can be written in a Hamiltonian form

$$u_t = \mathcal{H}\left(\frac{\delta \rho}{\delta u}\right),$$

where ρ is a conserved density and $\mathcal{H}$ is a Hamiltonian operator. It is known that this operator satisfies the equation

$$(D_t - F_*)\,\mathcal{H} = \mathcal{H}(D_t + F_*^t), \tag{0.6}$$

which means that $\mathcal{H}$ takes cosymmetries to symmetries. Besides (0.6) the Hamiltonian operator should satisfy certain identities (see for example, [1]) equivalent to the skew-symmetricity and the Jacobi identity for the corresponding Poisson bracket. It is easy to see that the ratio $\mathcal{H}_2\mathcal{H}_1^{-1}$ of any two Hamiltonian operators is a recursion operator.

As the rule, the Hamiltonian operators are local (i.e. differential) or quasilocal operators. The latter means that

$$\mathcal{H} = H + \sum_{i=1}^{m} G_i D_x^{-1} \bar{G}_i, \tag{0.7}$$

where H is a differential operator and $G_i, \bar{G}_i$ are fixed symmetries. It is clear that acting by the operator (0.7) on any cosymmetry, we get a local symmetry.

For example, for the Korteweg-de Vries equation

$$u_t = u_{xxx} + 6\,u\,u_x$$

the simplest recursion operator (see [3])

$$\mathcal{R} = D_x^2 + 4u + 2u_x D_x^{-1} \tag{0.8}$$

is quasilocal with $k = 1$, $G_1 = 2u_x$, and $g_1 = 1$. This operator is the ratio of two local Hamiltonian operators

$$\mathcal{H}_1 = D_x, \qquad \mathcal{H}_2 = D_x^3 + 4uD_x + 2u_x.$$

For the systematic investigation of quasilocal (weakly nonlocal in terminology of [4]) operators related to the Korteweg-de Vries and the nonlinear Schrödinger equations see [4, 5].

It is possible to prove that for the Korteweg-de Vries equation the associative algebra $\mathbf{A}$ of all quasilocal recursion operators is generated by operator (0.8). In other words, this algebra is isomorphic to the algebra of all polynomials in one variable.

Our main observation is that it is not true for such integrable models as the Krichever-Novikov and the Landau-Lifshitz equations. These equations play a role of the universal models for the classes of KdV-type and NLS-type equations. It appears that other integrable equations from these classes are limiting cases or can be linked to these models by differential substitutions [6, 7].

It turns out that for these models, known to be elliptic, the algebra **A** is isomorphic to the coordinate ring of the elliptic curve. Since the algebra **A** is defined in terms of the equation (0.1) only, this is an invariant way to associate a proper algebraic curve with any integrable equation and, in particular, to give a rigorous intrinsic definition of elliptic models.

This paper is organized as follows. In Section **1**, we consider the Krichever-Novikov equation [8], which is the simplest known one-field elliptic model. We show that there exist two quasilocal recursion operators of orders 4 and 6 related by the equation of elliptic curve. These recursion operators are ratios of the corresponding quasilocal Hamiltonian operators. The simplest quasilocal Hamiltonian operator of order -1

$$\mathcal{H}_0 = u_x D_x^{-1} u_x$$

for the Krichever-Novikov equation was found in [2]. From the results of this paper it follows that this equation has also a Hamiltonian operator of order 3. In Section **1** we present one more quasilocal Hamiltonian operator of the fifth order for the hierarchy of the Krichever-Novikov equation. It seems to be interesting to investigate this multi-Hamiltonian structure for the Krichever-Novikov equation in frames of the bi-Hamiltonian approach [9].

In Section **2** we obtain similar results for the Landau-Lifshitz equation. It turns out that for this model there exist quasilocal recursion operators of orders 2 and 3 related by an elliptic curve equation. The corresponding quasilocal Hamiltonian operators are also found.

In Sections **1,2** we have used the direct way to construct recursion and Hamiltonian operators based on the ansatzes (0.5) and (0.7). On the other hand there are several schemes that use $L-A$-pairs for this purpose. For example, there is a construction related to the squared eigenfunctions of the Lax operator L [10]-[12].

An alternative approach is based on the explicit formulas for the A- operators (see [13, 14, 15]). Some version of this approach has been suggested in [16]. In Section **3** we generalize the main idea of this work to the case of the Landau-Lifshitz equation. As a result, a deep relationship between the algebra **A** of quasilocal recursion operators and the algebra of multiplicators of the Lax structure becomes evident. In particular, the existence of two recursion operators related by elliptic curve equation follows from a similar property for multiplicators.

1 The Krichever-Novikov equation.

The Krichever-Novikov equation [8, 17] can be written in the form

$$u_{t_1} = u_{xxx} - \frac{3}{2}\frac{u_{xx}^2}{u_x} + \frac{P(u)}{u_x}, \qquad P^{(V)} = 0. \tag{1.9}$$

Denote by G_1 the right hand side of (1.9). The fifth order symmetry of (1.9) is given by

$$G_2 = u_5 - 5\frac{u_4 u_2}{u_1} - \frac{5}{2}\frac{u_3^2}{u_1} + \frac{25}{2}\frac{u_3 u_2^2}{u_1^2} - \frac{45}{8}\frac{u_2^4}{u_1^3} - \frac{5}{3}P\frac{u_3}{u_1^2} + \frac{25}{6}P\frac{u_2^2}{u_1^3} - \frac{5}{3}P'\frac{u_2}{u_1} - \frac{5}{18}\frac{P^2}{u_1^3} + \frac{5}{9}u_1 P''.$$

The simplest three conserved densities of (1.9) are

$$\rho_1 = -\frac{1}{2}\frac{u_2^2}{u_1^2} - \frac{1}{3}\frac{P}{u_1^2}, \qquad \rho_2 = \frac{1}{2}\frac{u_3^2}{u_1^2} - \frac{3}{8}\frac{u_2^4}{u_1^4} + \frac{5}{6}P\frac{u_2^2}{u_1^4} + \frac{1}{18}\frac{P^2}{u_1^4} - \frac{5}{9}P'',$$

$$\begin{aligned}\rho_3 = {} & \frac{u_4^2}{u_1^2} + 3\frac{u_3^3}{u_1^3} - \frac{19}{2}\frac{u_3^2u_2^2}{u_1^4} + \frac{7}{3}P\frac{u_3^2}{u_1^4} + \frac{35}{9}P'\frac{u_2^3}{u_1^4} + \frac{45}{8}\frac{u_2^6}{u_1^6} - \frac{259}{36}\frac{u_2^4P}{u_1^6} + \frac{35}{18}P^2\frac{u_2^2}{u_1^6} \\ & -\frac{14}{9}P''\frac{u_2^2}{u_1^2} + \frac{1}{27}\frac{P^3}{u_1^6} - \frac{14}{27}\frac{P''P}{u_1^2} - \frac{7}{27}\frac{P'^2}{u_1^2} - \frac{14}{9}P^{(IV)}u_1^2.\end{aligned}$$

In the paper [2] the forth order quasilocal recursion operator of the form

$$\mathcal{R}_1 = D_x^4 + a_1D_x^3 + a_2D_x^2 + a_3D_x + a_4 + G_1D_x^{-1}\frac{\delta\rho_1}{\delta u} + u_xD_x^{-1}\frac{\delta\rho_2}{\delta u},$$

was found. Here the coefficients a_i are given by

$$\begin{aligned} a_1 &= -4\frac{u_2}{u_1}, \qquad a_2 = 6\frac{u_2^2}{u_1^2} - 2\frac{u_3}{u_1} - \frac{4}{3}\frac{P}{u_1^2}, \\ a_3 &= -2\frac{u_4}{u_1} + 8\frac{u_3u_2}{u_1^2} - 6\frac{u_2^3}{u_1^3} + 4P\frac{u_2}{u_1^3} - \frac{2}{3}\frac{P'}{u_1}, \\ a_4 &= \frac{u_5}{u_1} - 2\frac{u_3^2}{u_1^2} + 8\frac{u_3u_2^2}{u_1^3} - 4\frac{u_4u_2}{u_1^2} - 3\frac{u_2^4}{u_1^4} + \frac{4}{9}\frac{P^2}{u_1^4} + \frac{4}{3}P\frac{u_2^2}{u_1^4} + \frac{10}{9}P'' - \frac{8}{3}P'\frac{u_2}{u_1^2}.\end{aligned}$$

The following statement can be verified directly.

Theorem 1. *There exists one more quasilocal recursion operator for* (1.9) *of the form*

$$\begin{aligned}\mathcal{R}_2 = {} & D_x^6 + b_1D_x^5 + b_2D_x^4 + b_3D_x^3 + b_4D_x^2 + b_5D_x + b_6 - \frac{1}{2}u_xD_x^{-1}\frac{\delta\rho_3}{\delta u} \\ & + G_1D_x^{-1}\frac{\delta\rho_2}{\delta u} + G_2D_x^{-1}\frac{\delta\rho_1}{\delta u},\end{aligned}$$

where

$$
\begin{aligned}
b_1 &= -6\frac{u_2}{u_1}, \qquad b_2 = -9\frac{u_3}{u_1} - 2\frac{P}{u_1^2} + 21\frac{u_2^2}{u_1^2},\\
b_3 &= -11\frac{u_4}{u_1} + 60\frac{u_3u_2}{u_1^2} + 14P\frac{u_2}{u_1^3} - 57\frac{u_2^3}{u_1^3} - 3\frac{P'}{u_1},\\
b_4 &= -4\frac{u_5}{u_1} + 38\frac{u_4u_2}{u_1^2} + 22\frac{u_3^2}{u_1^2} + 99\frac{u_2^4}{u_1^4} - 155\frac{u_3u_2^2}{u_1^3} + \frac{34}{3}P\frac{u_3}{u_1^3} - 44P\frac{u_2^2}{u_1^4}\\
&\quad + \frac{4}{3}\frac{P^2}{u_1^4} + 12P'\frac{u_2}{u_1^2} - P'',\\
b_5 &= -2\frac{u_6}{u_1} + 29\frac{u_4u_3}{u_1^2} + 80P\frac{u_2^3}{u_1^5} + \frac{23}{3}P'\frac{u_3}{u_1^2} - 104\frac{u_2u_3^2}{u_1^3} - 70\frac{u_4u_2^2}{u_1^3} + 241\frac{u_2^3u_3}{u_1^4} + 14\frac{u_5u_2}{u_1^2}\\
&\quad + \frac{20}{3}P\frac{u_4}{u_1^3} - \frac{170}{3}P\frac{u_2u_3}{u_1^4} + \frac{4}{3}\frac{P'P}{u_1^3} - 22P'\frac{u_2^2}{u_1^3} + 2P''\frac{u_2}{u_1} - \frac{16}{3}P^2\frac{u_2}{u_1^5} - 108\frac{u_2^5}{u_1^5},\\
b_6 &= \frac{u_7}{u_1} - 6\frac{u_2u_6}{u_1^2} + \frac{8}{9}P^2\frac{u_2^2}{u_1^6} - 195\frac{u_3^2u_2^2}{u_1^4} + 6P\frac{u_3^2}{u_1^4} + \frac{142}{3}P\frac{u_2^4}{u_1^6} + \frac{28}{9}P'P\frac{u_2}{u_1^4} + 101\frac{u_4u_3u_2}{u_1^3}\\
&\quad + \frac{34}{3}P\frac{u_4u_2}{u_1^4} - 72\frac{u_2^6}{u_1^6} - \frac{28}{9}P'''u_2 + \frac{38}{3}P''\frac{u_2^2}{u_1^2} - \frac{19}{3}P'\frac{u_4}{u_1^2} - \frac{122}{3}P'\frac{u_2^3}{u_1^4} - 10\frac{u_4^2}{u_1^2} + 22\frac{u_3^3}{u_1^3}\\
&\quad - \frac{178}{3}P\frac{u_3u_2^2}{u_1^5} + \frac{14}{9}P^{(IV)}u_1^2 + \frac{113}{3}P'\frac{u_3u_2}{u_1^3} - \frac{2}{3}P\frac{u_5}{u_1^3} - \frac{17}{3}P''\frac{u_3}{u_1} - \frac{4}{3}P^2\frac{u_3}{u_1^5} - 89\frac{u_4u_2^3}{u_1^4}\\
&\quad + 236\frac{u_3u_2^4}{u_1^5} - 13\frac{u_5u_3}{u_1^2} + 25\frac{u_5u_2^2}{u_1^3} - \frac{7}{9}\frac{P'^2}{u_1^2} - \frac{8}{27}\frac{P^3}{u_1^6} - \frac{4}{9}\frac{P''P}{u_1^2}.
\end{aligned}
$$

The operators $\mathcal{R}_1$ and $\mathcal{R}_2$ are related by the following elliptic curve

$$
\mathcal{R}_2^2 = \mathcal{R}_1^3 - \phi\mathcal{R}_1 - \theta, \tag{1.10}
$$

where

$$
\begin{aligned}
\phi &= \frac{16}{27}\Big((P'')^2 - 2P'''P' + 2P^{(IV)}P\Big),\\
\theta &= \frac{128}{243}\Big(-\frac{1}{3}(P'')^3 - \frac{3}{2}(P')^2P^{(IV)} + P'P''P''' + 2P^{(IV)}P''P - P(P''')^2\Big).
\end{aligned}
$$

Remark 1. The relation (1.10) is understood as an identity in the non-commutative field of pseudo-differential series [18] of the form

$$
A = a_m D_x^m + a_{m-1}D_x^{m-1} + \cdots + a_0 + a_{-1}D_x^{-1} + a_{-2}D_x^{-2} + \cdots .
$$

Here a_i are local functions and the multiplication is defined by

$$
aD_x^k \circ bD_x^m = a\left(bD_x^{m+k} + C_k^1 D_x(b)D_x^{k+m-1} + C_k^2 D_x^2(b)D_x^{k+m-2} + \cdots\right),
$$

where $k, m \in \mathbb{Z}$ and

$$C_n^j = \frac{n(n-1)(n-2)\cdots(n-j+1)}{j!}.$$

Remark 2. It is easy to verify that ϕ and θ are constants for any polynomial $P(u)$, where $\deg P \le 4$. Under Möbius transformations of the form

$$u \to \frac{\alpha u + \beta}{\gamma u + \delta}$$

in equation (1.9) the polynomial $P(u)$ changes according to the same rule as in the differential $\omega = \dfrac{du}{\sqrt{P(u)}}$. The expressions ϕ and θ are invariants with respect to the Möbius group action.

Remark 3. Of course, the ratio $\mathcal{R}_3 = \mathcal{R}_2\mathcal{R}_1^{-1}$ satisfies equation (0.4). It belongs to the noncommutative field of differential operator fractions [19]. Any element of this field can be written in the form $P_1 P_2^{-1}$ for some differential operators P_i. So, according to our definition, $\mathcal{R}_3$ is a recursion operator of order 2. However, this operator is not quasilocal and it is unclear how to apply it even to the simplest commuting flow u_x.

The recursion operators presented above appear to be ratios

$$\mathcal{R}_1 = \mathcal{H}_1\mathcal{H}_0^{-1}, \qquad \mathcal{R}_2 = \mathcal{H}_2\mathcal{H}_0^{-1}$$

of the following quasilocal Hamiltonian operators

$$\begin{aligned}
\mathcal{H}_1 &= \tfrac{1}{2}(u_x^2 D_x^3 + D_x^3 u_x^2) + (2u_{xxx}u_x - \tfrac{9}{2}u_{xx}^2 - \tfrac{2}{3}P)D_x + D_x(2u_{xxx}u_x - \tfrac{9}{2}u_{xx}^2 - \tfrac{2}{3}P)\\
&\quad + G_1 D_x^{-1} G_1 + u_x D_x^{-1} G_2 + G_2 D_x^{-1} u_x,\\
\mathcal{H}_2 &= \tfrac{1}{2}(u_x^2 D_x^5 + D_x^5 u_x^2) + (3u_{xxx}u_x - \tfrac{19}{2}u_{xx}^2 - P)D_x^3 + D_x^3(3u_{xxx}u_x - \tfrac{19}{2}u_{xx}^2 - P)\\
&\quad + hD_x + D_x h + G_1 D_x^{-1} G_2 + G_2 D_x^{-1} G_1 + u_x D_x^{-1} G_3 + G_3 D_x^{-1} u_x,
\end{aligned}$$

where

$$h = u_{xxxxx}u_x - 9u_{xxxx}u_{xx} + \frac{19}{2}u_{xxx}^2 - \frac{2}{3}\frac{u_{xxx}}{u_x}(5P - 39u_{xx}^2) + \frac{u_{xx}^2}{u_x^2}(5P - 9u_{xx}^2) + \frac{2}{3}\frac{P^2}{u_x^2} + u_x^2 P'',$$

and $G_3 = \mathcal{R}_1(G_1) = \mathcal{R}_2(u_x)$ is the seventh order symmetry of (1.9):

$$\begin{aligned}
G_3 &= u_7 - 7\frac{u_2u_6}{u_1} - \frac{7}{6}\frac{u_5}{u_1^2}(2P + 12u_3u_1 - 27u_2^2) - \frac{21}{2}\frac{u_4^2}{u_1} + \frac{21}{2}\frac{u_4}{u_1^3}u_2(2P - 11u_2^2)\\
&\quad - \frac{7}{3}\frac{u_4}{u_1^2}(2P'u_1 - 51u_2u_3) + \frac{49}{2}\frac{u_3^3}{u_1^2} + \frac{7}{12}\frac{u_3^2}{u_1^3}(22P - 417u_2^2) + \frac{2499}{8}\frac{u_2^4}{u_1^4}u_3\\
&\quad + \frac{91}{3}P'\frac{u_2}{u_1^2}u_3 - \frac{595}{6}P\frac{u_2^2}{u_1^4}u_3 - \frac{35}{18}\frac{u_3}{u_1^4}(2P''u_1^4 - P^2) - \frac{1575}{16}\frac{u_2^6}{u_1^5} + \frac{1813}{24}\frac{u_2^4}{u_1^5}P\\
&\quad - \frac{203}{6}\frac{u_2^3}{u_1^3}P' + \frac{49}{36}\frac{u_2^2}{u_1^5}(6P''u_1^4 - 5P^2) - \frac{7}{9}\frac{u_2}{u_1^3}(2P'''u_1^4 - 5PP') + \frac{7}{54}\frac{P^3}{u_1^5}\\
&\quad - \frac{7}{9}P''\frac{P}{u_1} + \frac{7}{9}P''''u_1^3 - \frac{7}{18}\frac{P'^2}{u_1}.
\end{aligned}$$

2 The Landau-Lifshitz equation.

In this section we consider the Landau-Lifshitz equation written in the form

$$\begin{aligned} u_{t_2} &= -u_{xx} + 2\psi\,(u_x^2 - P(u)) + \tfrac{1}{2}P'(u) \\ v_{t_2} &= \quad v_{xx} + 2\psi\,(v_x^2 - P(v)) - \tfrac{1}{2}P'(v), \end{aligned} \tag{2.11}$$

where

$$\psi = (u - v)^{-1},$$

and P is an arbitrary fourth degree polynomial. The usual vectorial form of Landau-Lifshitz equation gives rise to a system of the form (2.11) after the stereographic projection (see Section **3** for details).

The third order symmetry of (2.11) is given by

$$\begin{aligned} u_{t_3} &= u_{xxx} - 6u_x u_{xx}\psi + 6u_x^3\psi^2 - \tfrac{1}{2}u_x P''(v) - 3u_x\psi P'(v) - 6u_x\psi^2 P(v), \\ v_{t_3} &= v_{xxx} + 6v_x v_{xx}\psi + 6v_x^3\psi^2 - \tfrac{1}{2}v_x P''(v) - 3v_x\psi P'(v) - 6v_x\psi^2 P(v). \end{aligned}$$

In case of the system of evolution equation (2.11) the symmetries and cosymmetries can be treated as two-dimensional vectors. We introduce the following notation for symmetries:

$$G_1 = (u_x, v_x)^t, \qquad G_2 = (u_{t_2}, v_{t_2})^t, \qquad G_3 = (u_{t_3}, v_{t_3})^t. \tag{2.12}$$

Thus G_{ij} is j-th component of the symmetry G_i. The simplest cosymmetries are given by

$$g_1 = \psi^2(v_x, -u_x)^t, \qquad g_2 = \psi^2(v_{t_2}, -u_{t_2})^t, \qquad g_3 = \psi^2(v_{t_3}, -u_{t_3})^t. \tag{2.13}$$

These cosymmetries are variational derivatives $(\frac{\delta\rho}{\delta u}, \frac{\delta\rho}{\delta v})^t$ of the following conserved densities

$$\begin{aligned} &\rho_1 = \tfrac{1}{2}\psi(u_x + v_x), \qquad \rho_2 = \psi^2(u_x v_x - P(v)) - \tfrac{1}{2}\psi P'(v) - \tfrac{1}{12}P''(v), \\ &\rho_3 = \tfrac{1}{2}\psi^2(u_x v_{xx} - u_{xx}v_x) + \psi^3 u_x v_x(u_x + v_x) - \tfrac{1}{2}u_x\psi(4\psi^2 P(u) + P''(u) - 3\psi P'(u)). \end{aligned}$$

The coefficients of operators $F_*, \mathcal{R}$, and $\mathcal{H}$ from (0.4), (0.6) are matrices. For the quasilocal recursion operators the non-local terms can be written as $AD_x^{-1}B^t$, where A and B are $k \times 2$ matrices whose columns are symmetries and cosymmetries, correspondingly. For quasilocal Hamiltonian operators the columns are symmetries for both A and B.

Theorem 2. *Equation* (2.11) *possesses the following quasilocal recursion operators*:

$$\mathcal{R}_1 = \begin{pmatrix} R^1_{11} & 0 \\ 0 & R^1_{22} \end{pmatrix} - 2\begin{pmatrix} G_{11} & G_{21} \\ G_{12} & G_{22} \end{pmatrix} D_x^{-1} \begin{pmatrix} g_{21} & g_{22} \\ g_{11} & g_{12} \end{pmatrix} \tag{2.14}$$

and

$$\mathcal{R}_2 = \begin{pmatrix} R^2_{11} & R^2_{12} \\ R^2_{21} & R^2_{22} \end{pmatrix} + 2\begin{pmatrix} G_{11} & G_{21} & G_{31} \\ G_{12} & G_{22} & G_{32} \end{pmatrix} D_x^{-1} \begin{pmatrix} g_{31} & g_{32} \\ g_{21} & g_{22} \\ g_{11} & g_{12} \end{pmatrix}, \tag{2.15}$$

where G_{ij} and g_{ij} are defined by (2.12), (2.13), *and*

$$
\begin{aligned}
R^1_{11} &= D_x^2 - 4\psi u_x D_x + 2\psi^2 u_x(v_x+3u_x) - 2\psi u_{xx} - \tfrac{1}{3}P''(v) - 4P(v)\psi^2 - 2\psi P'(v),\\
R^1_{22} &= D_x^2 + 4\psi v_x D_x + 2\psi^2 v_x(u_x+3v_x) + 2\psi v_{xx} - \tfrac{1}{3}P''(v) - 4P(v)\psi^2 - 2\psi P'(v),\\
R^2_{11} &= D_x^3 - 6\psi u_x D_x^2 + (6\psi^2(3u_x^2 - P(v)) - 6\psi u_{xx} - \tfrac{1}{2}P''(v) - 3\psi P'(v))D_x + \psi(P''(v)u_x - 2u_{xxx})\\
&\quad +4\psi^3 u_x(6P(v) - 6u_x^2 - v_x^2 - u_x v_x) + \psi^2(2v_x u_{xx} - 2u_x v_{xx} + 9P'(v)u_x + 18u_x u_{xx}),\\
R^2_{12} &= 2\psi^2(P(u) - u_x^2)D_x - 4\psi^3(P(u)(u_x - v_x) + u_x^2 v_x - u_x^3) + \psi^2 u_x(P'(u) - 2u_{xx}),\\
R^2_{21} &= -2\psi^2(P(v) - v_x^2)D_x + 4\psi^3(P(v)(u_x - v_x) - v_x^2 u_x + v_x^3) - \psi^2 v_x(P'(v) - 2v_{xx}),\\
R^2_{22} &= -D_x^3 - 6v_x\psi D_x^2 + (6\psi^2(P(v) - 3v_x^2) - 6v_{xx}\psi + 3\psi P'(v) + \tfrac{1}{2}P''(v))D_x\\
&\quad -4\psi^3 v_x(6v_x^2 + v_x u_x + u_x^2) + 2\psi^2(3v_x P'(v) - 9v_x v_{xx} - u_x v_{xx} + v_x u_{xx} - \tfrac{3}{2}v_x P'(u))\\
&\quad +12\psi^3 v_x(P(v) + P(u)) + \psi(v_x P''(v) - 2v_{xxx}).
\end{aligned}
$$

Operators $\mathcal{R}_1$ and $\mathcal{R}_2$ are related by the following elliptic curve equation

$$\mathcal{R}_2^2 - \mathcal{R}_1^3 - \varphi\mathcal{R}_1 - \vartheta E = 0, \tag{2.16}$$

where E stands for the unity matrix, and

$$
\begin{aligned}
\varphi &= 12\psi^4 P(v)(P(u) - P(v)) + 4\psi^3(P'(v)P(u) - P(v)P'(u) - 3P(v)P'(v))\\
&\quad -\psi^2(2P''(v)P(v) + P'(v)P'(u) + 3P'(v)^2) - \psi P''(v)P'(v) - \tfrac{1}{12}P''(v)^2,
\end{aligned}
$$

$$
\begin{aligned}
\vartheta &= 16\psi^6 P(v)(P(v) - P(u))^2 + 8\psi^5(P(v) - P(u))(P(v)P'(u) + 3P'(v)P(v))\\
&\quad +\psi^4(P(v)P'(u)^2 - 3P'(v)^2P(u) + 6P(v)(P'(u)P'(v) + 2P'(v)^2) + 4P(v)P''(v)(P(v) - P(u)))\\
&\quad +\psi^2 P''(v)(P'(v)^2 + \tfrac{1}{3}P''(v)P(v) + \tfrac{1}{6}P'(v)P'(u)) + \tfrac{1}{6}\psi P''(v)^2P'(v) + \tfrac{1}{108}P''(v)^3\\
&\quad +\psi^3(P'(v)^2P'(u) + \tfrac{2}{3}P''(v)(P(v)P'(u) - P'(v)P(u)) + 4P''(v)P(v)P'(v) + 2P'(v)^3).
\end{aligned}
$$

It is easy to verify that φ and ϑ are constants for any polynomial P, if $\deg P \le 4$. They are invariants with respect to the Möbius group action just as in the case of the Krichever-Novikov equation (see Remark 2).

Remark 4. Possibly the odd recursion operator found in [20] is a rational function of recursion operators from Theorem **2**.

It turns out that the above recursion operators are ratios

$$\mathcal{R}_1 = \mathcal{H}_1\mathcal{H}_0^{-1}, \qquad \mathcal{R}_2 = \mathcal{H}_2\mathcal{H}_0^{-1}$$

of the following quasilocal Hamiltonian operators

$$\mathcal{H}_0 = \begin{pmatrix} 0 & \frac{1}{\psi^2} \\ -\frac{1}{\psi^2} & 0 \end{pmatrix},$$

$$
\mathcal{H}_1 = \begin{pmatrix} 0 & \mathcal{H}^1_{12} \\ \mathcal{H}^1_{21} & 0 \end{pmatrix} - 2 \begin{pmatrix} G_{11} & G_{21} \\ G_{12} & G_{22} \end{pmatrix} D_x^{-1} \begin{pmatrix} G_{21} & G_{22} \\ G_{11} & G_{12} \end{pmatrix},
$$

$$
\mathcal{H}_2 = \begin{pmatrix} \mathcal{H}^2_{11} & \mathcal{H}^2_{12} \\ \mathcal{H}^2_{21} & \mathcal{H}^2_{22} \end{pmatrix} + 2 \begin{pmatrix} G_{11} & G_{21} & G_{31} \\ G_{12} & G_{22} & G_{32} \end{pmatrix} D_x^{-1} \begin{pmatrix} G_{31} & G_{32} \\ G_{21} & G_{22} \\ G_{11} & G_{12} \end{pmatrix},
$$

where

$$
\mathcal{H}^1_{12} = \frac{1}{\psi^2} D_x^2 - 4\frac{v_x}{\psi} D_x - 2\frac{v_{xx}}{\psi} + 2v_x^2 + 6u_x v_x - \frac{1}{3}\frac{P''(v)}{\psi^2} - 2\frac{P'(v)}{\psi} - 4P(v),
$$

$$
\mathcal{H}^1_{21} = -\frac{1}{\psi^2} D_x^2 - 4\frac{u_x}{\psi} D_x - 2\frac{u_{xx}}{\psi} - 6u_x v_x - 2u_x^2 + \frac{1}{3}\frac{P''(v)}{\psi^2} + 2\frac{P'(v)}{\psi} + 4P(v),
$$

$$
\mathcal{H}^2_{11} = 2(u_x^2 - P(u))D_x + 2u_x u_{xx} - P'(u)u_x, \qquad \mathcal{H}^2_{22} = 2(v_x^2 - P(v))D_x + 2v_x v_{xx} - P'(v)v_x,
$$

$$
\begin{aligned}
\mathcal{H}^2_{12} = {}& \frac{1}{\psi^2} D_x^3 - 6\frac{v_x}{\psi} D_x^2 + \left(12u_x v_x + 6\Big(v_x^2 - \frac{v_{xx}}{\psi} - P(v)\Big) - \frac{1}{2}\frac{P''(v)}{\psi^2} - 3\frac{P'(v)}{\psi}\right) D_x + v_x \frac{P''(v)}{\psi} \\
& -2\frac{v_{xxx}}{\psi} + 6v_x v_{xx} + 8v_x u_{xx} + 4u_x v_{xx} + 4\psi(u_x + v_x)(3P(v) - 4u_x v_x) + P'(v)(3u_x + 6v_x),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{H}^2_{21} = {}& \frac{1}{\psi^2} D_x^3 + 6\frac{u_x}{\psi} D_x^2 + \left(12u_x v_x + 6\Big(u_x^2 + \frac{u_{xx}}{\psi} - P(v)\Big) - \frac{1}{2}\frac{P''(v)}{\psi^2} - 3\frac{P'(v)}{\psi}\right) D_x \\
& +3v_x P'(u) + 2\frac{u_{xxx}}{\psi} - u_x \frac{P''(v)}{\psi} + 6u_x u_{xx} + 8u_x v_{xx} + 4v_x u_{xx} - 6P'(v)u_x \\
& \quad +16u_x v_x(u_x + v_x)\psi - 12\psi(P(v)u_x + P(u)v_x).
\end{aligned}
$$

3 Recursion operators and multiplicators.

The original vector form of Landau-Lifshitz reads as follows

$$
\mathbf{U}_t = \mathbf{U} \times \mathbf{U}_{xx} + \mathbf{U} \times J\mathbf{U}. \tag{3.17}
$$

Here $\mathbf{U} = (u_1, u_2, u_3)$, $|\mathbf{U}| = 1$, symbol $\times$ stands for the vector product, and $J = \mathrm{diag}(p, q, r)$ is an arbitrary constant diagonal matrix. The usual way to represent equation (3.17) as a two-component system is to make use of the stereographic projection. The transformation

$$
\mathbf{U} = (1 - uv, i + iuv, u + v)\,\psi, \quad \text{where} \quad i^2 = -1, \tag{3.18}
$$

coincides with the stereographic projection if we set $v = -1/\bar{u}$. This transformation takes equation (3.17) into the system of the form (2.11):

$$
\begin{aligned}
u_\tau &= -u_{xx} + 2\psi(u_x^2 - P(u)) + \tfrac{1}{2}P'(u), \\
v_\tau &= v_{xx} + 2\psi(v_x^2 - P(v)) - \tfrac{1}{2}P'(v),
\end{aligned} \tag{3.19}
$$

where $t = -i\tau$,

$$
P(u) = \frac{1}{4}(p - q)u^4 - \frac{1}{2}(p + q - 2r)u^2 + \frac{1}{4}(p - q).
$$

Here and below the expression $(\mathbf{A},\mathbf{B})$ denotes the standard Euclidean scalar product of the vectors $\mathbf{A}$ and $\mathbf{B}$. Note that for equation (3.19) curve (2.16) has the form

$$\mathcal{R}_2^2=\Big(\mathcal{R}_1+\frac{2p-q-r}{3}E\Big)\Big(\mathcal{R}_1+\frac{2q-p-r}{3}E\Big)\Big(\mathcal{R}_1+\frac{2r-p-q}{3}E\Big).$$

Recall the algebraic structure lying behind the elliptic Lax pair [21] for the Landau-Lifshitz equation. Let

$$\mathbf{e_1}=\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix},\quad \mathbf{e_2}=\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix},\quad \mathbf{e_3}=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}$$

be the standard basis in the Lie algebra $so(3)$. The Lax operator L for (3.17) is given by

$$L=D_x-\sum_{j=1}^{3}u_i\mathbf{E_i}, \tag{3.20}$$

where

$$\mathbf{E_1}=\frac{1}{\lambda}\mathbf{e_1}\sqrt{1-p\lambda^2},\qquad \mathbf{E_2}=\frac{1}{\lambda}\mathbf{e_2}\sqrt{1-q\lambda^2},\qquad \mathbf{E_3}=\frac{1}{\lambda}\mathbf{e_3}\sqrt{1-r\lambda^2}.$$

The operators A_i defining the Lax representations

$$L_{t_i}=[A_i,\,L] \tag{3.21}$$

for the commuting flows

$$\mathbf{U}_{t_i}=\mathbf{H}^{(i)}(\mathbf{U},\mathbf{U}_x,\ldots)$$

from the Landau-Lifshitz hierarchy belong to the Lie algebra $\mathcal{G}$ generated by $\mathbf{E_1},\mathbf{E_2},\mathbf{E_3}$.

Lemma 1. *The algebra $\mathcal{G}$ is spanned by*

$$\frac{1}{\lambda^{2i}}\mathbf{E_1},\quad \frac{1}{\lambda^{2i}}\mathbf{E_2},\quad \frac{1}{\lambda^{2i}}\mathbf{E_3},\quad \frac{1}{\lambda^{2i}}\bar{\mathbf{E}}_\mathbf{1},\quad \frac{1}{\lambda^{2i}}\bar{\mathbf{E}}_\mathbf{2},\quad \frac{1}{\lambda^{2i}}\bar{\mathbf{E}}_\mathbf{3},\qquad i=0,1,2,\ldots, \tag{3.22}$$

where

$$\bar{\mathbf{E}}_\mathbf{1}=\frac{1}{\lambda^2}\mathbf{e_1}\sqrt{1-q\lambda^2}\sqrt{1-r\lambda^2},\quad \bar{\mathbf{E}}_\mathbf{2}=\frac{1}{\lambda^2}\mathbf{e_2}\sqrt{1-p\lambda^2}\sqrt{1-r\lambda^2},\quad \bar{\mathbf{E}}_\mathbf{3}=\frac{1}{\lambda^2}\mathbf{e_3}\sqrt{1-p\lambda^2}\sqrt{1-q\lambda^2}.$$

Our main observation is that the recursion operators are in one-to one correspondence with the multiplicators (see [22, 23]) of the algebra $\mathcal{G}$.

Definition. A (scalar) function $\mu(\lambda)$ is called the *multiplicator* for the algebra $\mathcal{G}$ if $\mu(\lambda)\,\mathcal{G}\subset\mathcal{G}$. The order of pole of $\mu(\lambda)$ at $\lambda=0$ is called the order of the multiplicator.

It is easy to prove the following

Lemma 2. *The set of all multiplicators for $\mathcal{G}$ coincides with the polynomial ring generated by* 1,

$$\mu_1(\lambda) = \frac{1}{\lambda^2}, \qquad \mu_2(\lambda) = \frac{\sqrt{1-p\lambda^2}\sqrt{1-q\lambda^2}\sqrt{1-r\lambda^2}}{\lambda^3}.$$

It is clear that

$$\mu_2^2 = (\mu_1 - p)(\mu_1 - q)(\mu_1 - r).$$

Thus the ring of multiplicators is isomorphic to the coordinate ring of an elliptic curve.

The following construction establishes a correspondence between multiplicators and recursion operators. Let μ be a multiplicator of order $k > 0$. To find a relation between operators A_n and A_{n+k} we use the following anzats

$$A_{n+k} = \mu A_n + R_n, \qquad R_n \in \mathcal{G}, \qquad \operatorname{ord} R_n < k.$$

Substituting this into (3.21) and taking into account (3.20), we get

$$\sum_{j=1}^{3} H_j^{(n+k)} \mathbf{E_j} = \mu \sum_{j=1}^{3} H_j^{(n)} \mathbf{E_j} + \Big[\sum_{j=1}^{3} u_j \mathbf{E_j},\, R_n\Big] - \frac{dR_n}{dx}. \tag{3.23}$$

Both sides of this relation belong to $\mathcal{G}$. Equating in (3.23) the coefficients of basis elements (3.22), we find step by step unknown coefficients of R_n and eventually an expression for $H_j^{(n+k)}$ in terms of $H_j^{(n)}$ i.e. a recursion operator of order k.

For the simplest multiplicator $\mu_1 = \lambda^{-2}$, we have

$$R_n = \sum_{j=1}^{3} M_j \mathbf{E}_j + \sum_{j=1}^{3} F_j \bar{\mathbf{E}}_j.$$

It is easy to verify that (3.23) is equivalent to the following identities:

1. $\mathbf{H}^{(n)} - \mathbf{F} \times \mathbf{U} = 0$,
2. $\mathbf{F}_x + \mathbf{U} \times \mathbf{M} = 0$,
3. $\mathbf{H}^{(n+2)} = \mathbf{M}_x + \mathbf{F} \times J\mathbf{U}$,

where $\mathbf{M} = (M_1, M_2, M_3)$, $\mathbf{F} = (F_1, F_2, F_3)$. It follows from the first identity and the condition $|\mathbf{U}| = 1$ that

$$\mathbf{F} = \mathbf{U} \times \mathbf{H}^{(n)} + f\,\mathbf{U}. \tag{3.24}$$

To find f we note that the second identity implies $(\mathbf{U}, \mathbf{F}_x) = 0$. Substituting the expression (3.24) to this relation, we get

$$f = D_x^{-1}(\mathbf{U}, \mathbf{H}^{(n)} \times \mathbf{U}_x).$$

The second identity can be rewritten as

$$\mathbf{M} = \mathbf{U} \times \mathbf{F}_x + m\mathbf{U} = \mathbf{U}(\mathbf{U}, \mathbf{H}_x^{(n)}) - \mathbf{H}_x^{(n)} + f\mathbf{U} \times \mathbf{U}_x + m\mathbf{U}$$

for some function m. To find m, we substitute into the third identity the latter expression for $\mathbf{M}$ and take the scalar product by $\mathbf{U}$ of both sides of the relation thus obtained. Taking into account that $(\mathbf{H}^{(n+2)}, \mathbf{U}) = 0$, we get

$$m = D_x^{-1}\Big((J\mathbf{U}, \mathbf{H}^{(n)}) - (\mathbf{U}_x, \mathbf{H}_x^{(n)})\Big).$$

Now the third identity produces the following recursion operator for equation (3.17):

$$\begin{gathered}\mathbf{H}^{(n+2)} = (\mathbf{U}, \mathbf{H}_{xx}^{(n)})\mathbf{U} - \mathbf{H}_{xx}^{(n)} + (\mathbf{U}, \mathbf{H}_x^{(n)})\mathbf{U}_x - (\mathbf{U}, \mathbf{U}_x \times \mathbf{H}^{(n)})\mathbf{U} \times \mathbf{U}_x + (\mathbf{U}, J\mathbf{U})\mathbf{H}^{(n)} \\ -(\mathbf{U} \times \mathbf{U}_{xx} + \mathbf{U} \times J\mathbf{U})D_x^{-1}(\mathbf{U}, \mathbf{U}_x \times \mathbf{H}^{(n)}) - \mathbf{U}_x D_x^{-1}\left((\mathbf{U}_x, \mathbf{H}_x^{(n)}) - (J\mathbf{U}, \mathbf{H}^{(n)})\right).\end{gathered} \tag{3.25}$$

The recursion operator (3.25), along with the bi-Hamiltonian structure of the Landau-Lifshitz equation, was originally discovered in [24], whereas explicit formulas for higher symmetries were given earlier by Fuchssteiner [25].

For the second multiplicator μ_2 we set

$$R_n = \sum_{j=1}^{3} K_j \mathbf{E}_j + \sum_{j=1}^{3} M_j \bar{\mathbf{E}}_j + \frac{1}{\lambda^2}\sum_{j=1}^{3} F_j \mathbf{E}_j. \tag{3.26}$$

Upon the substitution of (3.26) to (3.23) we get the following identities

$$\begin{aligned} &1.\ \ \mathbf{H}^{(n)} = \mathbf{F} \times \mathbf{U}, \\ &2.\ \ \mathbf{F}_x = \mathbf{M} \times \mathbf{U}, \\ &3.\ \ \mathbf{M}_x - J\mathbf{H}^{(n)} = \mathbf{K} \times \mathbf{U}, \\ &4.\ \ \mathbf{H}^{(n+3)} = \mathbf{K}_x + \mathbf{M} \times J\mathbf{U}. \end{aligned} \tag{3.27}$$

The first two relations in (3.27) are analogous to (3), and therefore

$$\mathbf{F} = \mathbf{U} \times \mathbf{H}^{(n)} + f\mathbf{U}, \qquad f = D_x^{-1}(\mathbf{U}, \mathbf{H}^{(n)} \times \mathbf{U}_x),$$

$$\mathbf{M} = (\mathbf{U}, \mathbf{H}_x^{(n)})\mathbf{U} - \mathbf{H}_x^{(n)} + f\mathbf{U} \times \mathbf{U}_x + m\mathbf{U}.$$

It follows from $(3.27)_3$ that

$$\begin{gathered}\mathbf{K} = \mathbf{U} \times (\mathbf{M}_x - J\mathbf{H}^{(n)}) + k\mathbf{U} = (\mathbf{U}, \mathbf{H}_x^{(n)})\mathbf{U} \times \mathbf{U}_x - \mathbf{U} \times J\mathbf{H}^{(n)} - \mathbf{U} \times \mathbf{H}_{xx}^{(n)} \\ -\mathbf{U}_x(\mathbf{U}, \mathbf{H}^{(n)} \times \mathbf{U}_x) - ((\mathbf{U}_x, \mathbf{U}_x)\mathbf{U} + \mathbf{U}_{xx})f + m\mathbf{U} \times \mathbf{U}_x + \kappa\mathbf{U}.\end{gathered}$$

Functions m and κ can be found from the conditions

$$(\mathbf{U}, \mathbf{M}_x - J\mathbf{H}^{(n)}) = 0, \qquad (\mathbf{U}, \mathbf{H}^{(n+3)}) = 0,$$

which yield

$$m = D_x^{-1}((\mathbf{U}, J\mathbf{H}^{(n)}) - (\mathbf{U}_x, \mathbf{H}_x^{(n)})),$$
$$\kappa = D_x^{-1}((\mathbf{U}, \mathbf{U}_x \times J\mathbf{H}^{(n)}) - (\mathbf{U}, \mathbf{H}_{xx}^{(n)} \times \mathbf{U}_x) - \tfrac{1}{2}(\mathbf{U}, \mathbf{H}^{(n)} \times \mathbf{U}_x)((\mathbf{U}, J\mathbf{U}) + (\mathbf{U}_x, \mathbf{U}_x))$$
$$+(\mathbf{U}, \mathbf{H}_x^{(n)} \times J\mathbf{U})) - \tfrac{1}{2}((\mathbf{U}_x, \mathbf{U}_x) - (\mathbf{U}, J\mathbf{U}))D_x^{-1}(\mathbf{U}, \mathbf{H}^{(n)} \times \mathbf{U}_x).$$

Thus the second recursion operator reads as

$$\begin{aligned}
\mathbf{H}^{(n+3)} = {} & \mathbf{H}_{xxx}^{(n)} \times \mathbf{U} + \mathbf{H}_{xx}^{(n)} \times \mathbf{U}_x + (\mathbf{H}_{xx}^{(n)}, \mathbf{U})\mathbf{U} \times \mathbf{U}_x - (\mathbf{U}, \mathbf{H}_{xx}^{(n)} \times \mathbf{U}_x)\mathbf{U} \\
& +(\mathbf{H}_x^{(n)}, \mathbf{U})\mathbf{U} \times J\mathbf{U} + (\mathbf{H}_x^{(n)}, \mathbf{U})\mathbf{U} \times \mathbf{U}_{xx} - \mathbf{H}_x^{(n)} \times J\mathbf{U} + (\mathbf{U}, \mathbf{H}_x^{(n)} \times J\mathbf{U})\mathbf{U} \\
& -(\mathbf{U}, \mathbf{H}_x^{(n)} \times \mathbf{U}_x)\mathbf{U}_x - 2(\mathbf{U}, \mathbf{H}^{(n)} \times \mathbf{U}_x)(\mathbf{U}_x, \mathbf{U}_x)\mathbf{U} - \mathbf{U}_x \times J\mathbf{H}^{(n)} - \mathbf{U} \times J\mathbf{H}_x^{(n)} \\
& +(\mathbf{U}, \mathbf{U}_x \times J\mathbf{H}^{(n)})\mathbf{U} + (J\mathbf{H}^{(n)}, \mathbf{U})\mathbf{U} \times \mathbf{U}_x - (\mathbf{U}, \mathbf{H}^{(n)} \times \mathbf{U}_{xx})\mathbf{U}_x \\
& -2(\mathbf{U}, \mathbf{H}^{(n)} \times \mathbf{U}_x)\mathbf{U}_{xx} - (\mathbf{U} \times \mathbf{U}_{xx} + \mathbf{U} \times J\mathbf{U})D_x^{-1}\left((\mathbf{H}_x^{(n)}, \mathbf{U}_x) - (J\mathbf{H}^{(n)}, \mathbf{U})\right) \\
& - \left((\mathbf{U}_{xx} + \tfrac{3}{2}\mathbf{U}(\mathbf{U}_x, \mathbf{U}_x))_x - \tfrac{3}{2}\mathbf{U}_x(\mathbf{U}, J\mathbf{U})\right) D_x^{-1}(\mathbf{U}, \mathbf{H}^{(n)} \times \mathbf{U}_x) \\
& +\mathbf{U}_x D_x^{-1}\Big((\mathbf{U}, \mathbf{U}_x \times J\mathbf{H}^{(n)}) - (\mathbf{U}, \mathbf{H}_{xx}^{(n)} \times \mathbf{U}_x) - \tfrac{1}{2}(\mathbf{U}, \mathbf{H}^{(n)} \times \mathbf{U}_x)(\mathbf{U}, J\mathbf{U}) \\
& \quad -\tfrac{1}{2}(\mathbf{U}, \mathbf{H}^{(n)} \times \mathbf{U}_x)(\mathbf{U}_x, \mathbf{U}_x) + (\mathbf{H}_x^{(n)} \times J\mathbf{U}, \mathbf{U})\Big).
\end{aligned} \tag{3.28}$$

One can verify that under transformation (3.18) the operators defined by formulas (3.25) and (3.28) become $\frac{1}{3}(p+q+r)E - \mathcal{R}_1$ and $\mathcal{R}_2$ correspondingly.

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